Chapter 20
High-Pass and Low-Pass Filters

Two resistors, connected as in Fig. 20-1, make up a voltage divider circuit. If the input voltage is Vin, the output voltage Vout is expressed as

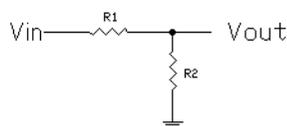$$V_{out} = V_{in} \times \frac{R2}{R1 + R2}$$

Fig. 20-1. Resistive voltage divider

This is a simple equation, but when the resistors are numbered differently, or if they are replaced by some other component, it is easy to confuse what is divided by what. A better way to express the equation is

$$V_{out} = V_{in} \times \frac{\text{resistance across the output}}{\text{resistance across the input}}$$

Alternatively, we might define the gain G as

$$Gain = \frac{V_{out}}{V_{in}} = \frac{\text{resistance across the output}}{\text{resistance across the input}}$$

In this case, the output appears across resistor R_2, while the input is applied to the series combination of R_1 and R_2, so we divide R_2 by $R_1 + R_2$. For example, if both resistors are 1000 ohms, then the gain would be

$$Gain = \frac{1000}{1000 + 1000} = \frac{1000}{2000} = \frac{1}{2}$$

RC Filters

Fig. 20-2 shows two other circuits which are configured the same way, but which substitute a capacitor instead of one of the resistors. We still use the same equation, except that we substitute the word impedance instead of resistance, like this:

$$Gain = \frac{V_{out}}{V_{in}} = \frac{\text{impedance across the output}}{\text{impedance across the input}} \qquad (20\text{-}1)$$
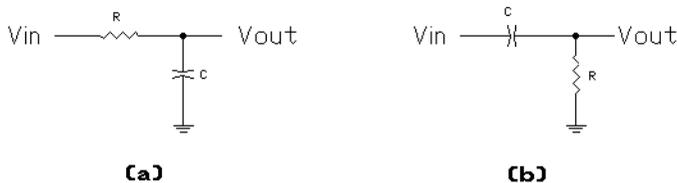

Fig. 20-2. (a) Low-pass and (b) High-pass RC filters

Circuit (a) is called an RC low-pass filter; RC because it uses a resistor and capacitor, and low-pass because it passes (lets through) low-frequency AC signals, but reduces or stops high-frequency signals. Circuit (b) is called a high-pass RC filter because it lets through high frequency signals, but reduces or stops low-frequency signals.

The reason that these circuits react to the frequency of the signal is because the reactance X_C of a capacitor depends on the frequency f:

$$X_C = \frac{1}{2\pi f\, C}$$

In simple terms, at high frequencies X_C is very small and so the capacitor behaves like a short circuit. At low frequencies, on the other hand, X_C is very large, and so the capacitor behaves like an open circuit.

At high frequencies the capacitor in the low-pass filter of Fig. 20-2 (a) looks like a short circuit, and therefore shorts the output signal, not letting high frequencies through. In the high-pass filter of Fig. 20-2 (b), on the other hand, the capacitor at low frequencies acts like an open circuit and does not let the signal go from the input to the output.

Much more interesting, though, is what happens at middle frequencies. Since X_C depends on the frequency, getting very large at one extreme and very small at the other, there has to be some frequency at which X_C is exactly equal to the resistance R. That is,

$$X_C = \frac{1}{2\pi f\, C} = R$$

We can find this frequency by simply solving this equation for the frequency:

$$f = \frac{1}{2\pi\, R\, C}$$

This particular frequency is usually called the *corner frequency*, *cutoff frequency*, or *3-db frequency*, for reasons we shall see shortly, and it is usually written f_C.

Let's look at a simple example. Suppose at the corner frequency, R is 1000 ohms, and X_C is also 1000 ohms. Back in Fig. 20-1, we saw that, when the two resistors have the same 1000-ohm resistance, the gain is $\frac{1}{2}$. If the impedances of the components in the two filters in Fig. 20-2 are also equal to 1000 ohms, what is the gain now?

If you guessed $\frac{1}{2}$, guess again. The actual gain at the corner frequency turns out to be 1 divided by $\sqrt{2}$, or 0.707, not $\frac{1}{2}$. To see the reason, we must return to equation (20-1). The impedance across the output is 1000 ohms in both cases, but you can't just add the 1000 ohms for the resistor to the 1000 ohms for the capacitor to get a 2000-ohm impedance across the input! When a
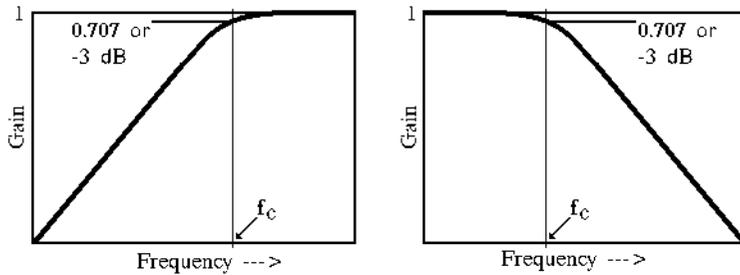
Fig. 20-3. High-pass (left) and Low-Pass (right) responses

situation is the opposite for the low-pass filter on the right — the gain is 1 at low frequencies, drops to 0.707 as we go up to the corner frequency, and then keeps dropping even farther as we go to higher and higher frequencies. The two kinds of filters are thus similar in a way, but opposite.

When the gain drops from 1 down to 0.707 at the corner frequency, this drop is equal to –3 decibels, so the corner frequency is also called the 3-db frequency.

resistance is in series with a reactance, we have to compute the impedance Z like this:

$$Z = \sqrt{R^2 + X^2}$$

Therefore the impedance across the input is

$$Z = \sqrt{1000^2 + 1000^2} = 1414 \text{ ohms}$$

and so

$$Gain = \frac{V_{out}}{V_{in}} = \frac{\text{impedance across the output}}{\text{impedance across the input}} = \frac{1000}{1414} = 0.707$$

(remember — this is only true at the corner frequency — this is the only frequency where X_C is exactly equal to R.)

Fig. 20-3 shows how the gain of these filters varies with frequency. For the high-pass filter at the left, the gain at very high frequencies is 1; it drops to 0.707 at the corner frequency f_C, and then keeps dropping lower and lower as we go to lower and lower frequencies. The
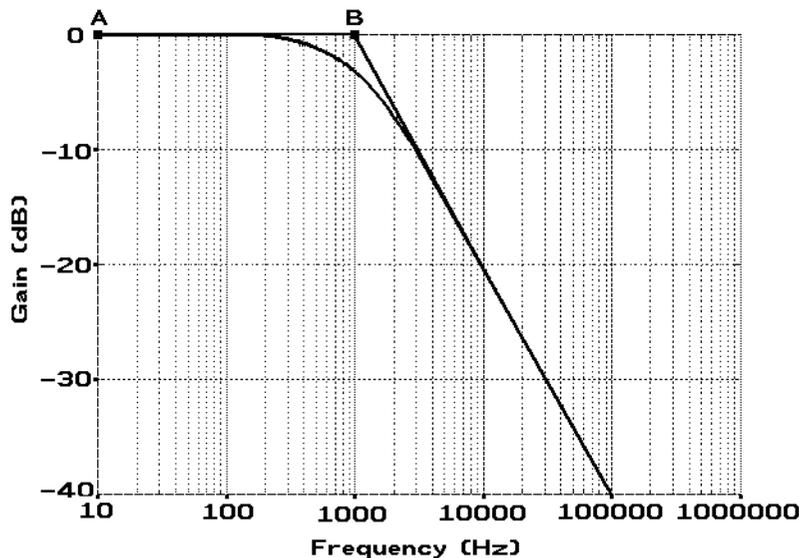
Example 20-1. Frequency Response Graphs

Given the values of the resistor and capacitor in a filter, it is easy to draw its frequency response graph, using a few shortcuts. Let's work through an example.

Suppose a low-pass filter circuit, as shown in Fig. 20-2 (a), uses a 1590-ohm resistor and 0.1 µF capacitor. The corner frequency is therefore

$$f_C = \frac{1}{2\pi R C} = \frac{1}{6.28 \times 1.59 \times 10^3 \times 10^{-7}}$$

$$= 1000 \text{ Hz}$$

Next, we need the right kind of graph paper. Linear graph paper — paper where the horizontal divisions are equally apart — won't do. We need special graph paper with a logarithmic frequency axis, as is shown in Fig. 20-4.

This particular graph paper contains five sections called *cycles*, but log graph paper can have more or fewer. Each cycle (a typical one is shown in Fig. 20-5) covers a 10-to-1 ratio, or *decade*, of frequency. Within that cycle, different frequency ranges use different amounts of space. Thus the distance from 10 to 20 Hz (an octave or 2-to-1 range) takes the same amount of

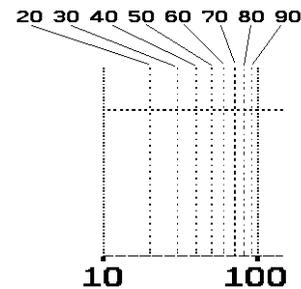Fig. 20-4. Detailed Low-Pass filter response

Fig. 20-5. One typical cycle

room as the octave from 20 to 40 Hz, or the octave from 40 to 80 Hz.

Each cycle is the same as any other. In this case, since the corner frequency is 1000 Hz, we want 1000 Hz to be somewhere in the center of the graph, so we labelled the cycles from 10 to 10,000,000 Hz. But we could have started at 1 Hz on the left, and stopped at 100,000 Hz at the right.

Next, we must label the vertical axis. Frequency responses are usually measured in decibels, so the vertical axis shows dB. Since plain RC filters have no gain, only loss, we do not need to plot any positive dB values, so we start with 0 dB at the top, and go downward, in this case down to -40 dB.

The next step is to draw two *asymptotes*. An asymptote is a straight line that the actual curve approaches, but never (theoretically) touches. In Fig. 20-4, one of these is the horizontal line from point A in the top left corner, over to point B. Point B is at 0 dB and 1000 Hz, the corner frequency.

The second asymptote starts at point B and slants down and to the right, forming a corner at point B — hence the name corner frequency. Although we know that this asymptote goes through point B, we must still decide how steeply to draw it. For simple filters such as these, the slope of this asymptote is always –6 dB per octave, or –20 dB per decade. Thus this line passes through –6 dB at 2000 Hz (one octave above 1000 Hz), passes through –20 dB at 10,000 Hz (one decade above the corner), and passes through –40 dB at 100,000 Hz (two decades above the corner frequency.)

The actual response curve (the heavy line in Fig. 20-4) approaches very close to the asymptotes at very low and very high frequencies (and is actually shown touching in the graph), but it passes below the actual corner. Fig. 20-6 shows a closeup of this region. The actual curve follows these rules:

- At the corner frequency, it is 3 dB below the corner.
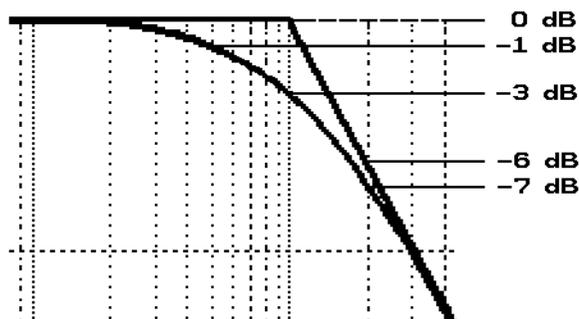- At half the corner frequency, and at double the corner frequency, i.e., an octave below and an

Fig. 20-6. Closeup of the corner

octave above, it is 1 dB below the asymptote. Note that the asymptote at 2000 Hz is at –6 dB, so the actual curve is 1 dB below that, at –7 dB.

- A decade below, and a decade above, it is almost right on the asymptote.

Note that the 1 dB and 3 dB values described here are only approximate; the actual values are slightly off, but engineers seldom quibble about a fraction of a dB. In any case, knowing how the curve behaves near the corner, it is fairly easy to fill in the actual curve freehand.

Three short comments before we continue:

(1) The above example dealt with a low-pass filter. The RC high-pass filter would be done exactly the same way, except that the graph would be a mirror image, as in Fig. 20-2 (a).

(2) This procedure only works well with logarithmic graph paper. If you try to use standard linear graph paper, the dropoff will not be a straight line, but will be curved. It is then hard to draw the right slope.

(3) The part of the curve that slopes downward is called the *skirt*.

Poles, Zeroes, and Filter Order

Both the high-pass and low-pass RC filters we have seen had skirts that slope down at 6 dB per octave, which is the same as 20 dB per decade. Although the corner frequency is also called a cutoff frequency, that term is not very accurate — there is no "cutoff" as such. The undesired frequencies are not cut off, but are merely reduced in the vicinity of the corner frequency. Even an octave away, they are only reduced by –7 dB, still almost half as large as at the maximum. You must go pretty far away from the cutoff frequency ("go far down the skirt") before the undesired signals are substantially reduced. Often this is just not good enough.

To make a significant difference, we need to increase the slope so that the response drops much faster as we go through and past the corner frequency. This requires more reactive components — capacitors or inductors — in the filter circuit. Each of these components increases the slope by 6 dB per octave (20 dB per decade), and the number of capacitors or inductors is called the *order* of the filter. For example, a second-order filter would drop off at 12 dB per octave, a third-order filter would drop off at 18 dB per octave, and so on. (Only inductors or capacitors that actually do filtering are counted — components that are in the circuit for other reasons, such as to keep out DC etc., don't count in the order.)

There are various ways to analyze or design filters and other circuits; one of these uses a mathematical method called the Laplace Transform, which is beyond the scope of this book. This method describes the per-
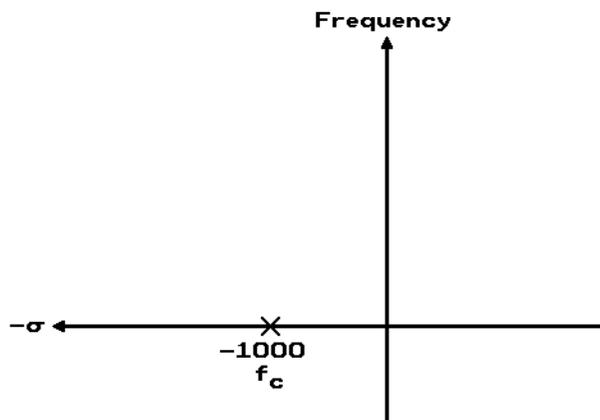
Fig. 20-7. The s-plane for a low-pass filter

Fig. 20-8. Calculating the gain at 2000 Hz

formance of a filter by its "poles" and "zeroes" — a pole is a place where a math expression becomes infinite, while a zero is a point where the expression becomes zero. For example, the expression

$$\frac{s}{s-2}$$

becomes infinite if s equals 2, because the denominator then becomes 0 and division by 0 is not allowed, so there is a pole at $s = 2$. On the other hand, the expression is zero if $s = 0$, so there is a zero at $s = 0$. We don't really need to know the full story about poles and zeroes, other than recognizing that their locations can be plotted in a diagram called the *s-plane*.

Each capacitor or inductor in a filter circuit introduces a pole into the Laplace equation for the gain, so a second-order filter can also be called a two-pole filter. In addition, some filters also have one or more zeroes.

Low-Pass RC filter graphical analysis

Fig. 20-7 shows the pole for the first-order RC low-pass filter in Example 20-1. The vertical axis in the s-plane is used to plot frequency in units of Hertz. The horizontal axis plots something called sigma (the Greek letter σ); we need not worry about what this means, except that it too is measured in Hertz. The single pole for Example 20-1 is shown as an × at -1000 on the sigma axis, since the corner frequency for the filter is at 1000 Hz, and the gain at any particular frequency is distance A, divided by distance D.

Fig. 20-8 shows how you would use this picture to calculate the gain of this filter at some frequency such as, for example, 2000 Hz. Begin by marking a point at 2000 Hz on the frequency (vertical) axis, using the same units as were used on the sigma)horizontal) axis. Then calculate the distance D from the Pythagorean formula:
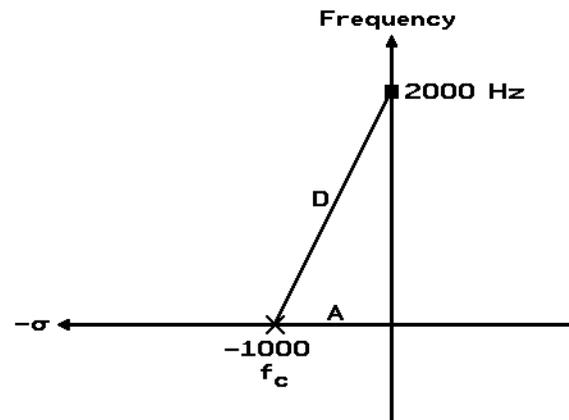
$$D = \sqrt{1000^2 + 2000^2} = 2236$$

The filter gain at this frequency is therefore

$$\text{Gain} = \frac{A}{D} = \frac{1000}{2236} = 0.447$$

Converting this to dB, we have

$$20 \times \log(0.4477) = -6.99 \text{ dB}$$

which is almost exactly the -7 dB we expect from our prior discussion at one octave above the corner frequency.

This method for calculating filter gain is very easy to automate with a spreadsheet program such as 123 or Excel, as in this sheet:

	A	B	C	D
1	Freq.	Dist. D	Gain A/D	dB
2	0	1000	1	0
3	500	1118	0.894	-0.97
4	1000	1414	0.707	-3.01
5	1500	1803	0.555	-5.12
6	2000	2236	0.447	-6.99
7	2500	2693	0.371	-8.60
8	3000	3162	0.316	-10.0
9	3500	3640	0.275	-11.2
10	4000	4123	0.243	-12.3
11	4500	4610	0.217	-13.3
12	5000	5099	0.196	-14.1

The formulas to enter for each column of the spreadsheet are:

Cell B2 = sqrt(A2^2 + 1000^2)

Cell C2 = 1000 / B2

Cell D2 = 20 * log(C2)
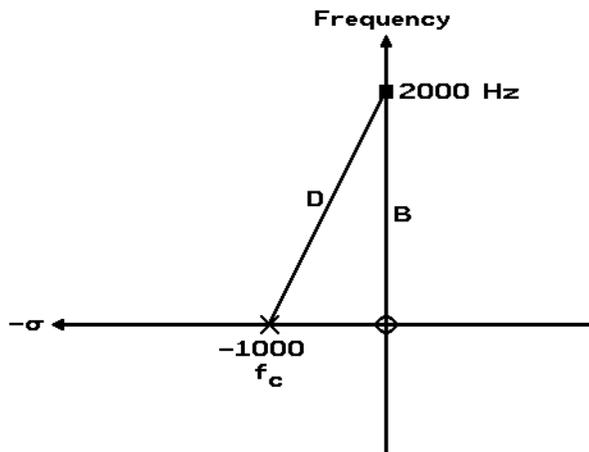
High-Pass RC filter graphical analysis

Fig. 20-9. The s-plane for a high-pass filter

The s-plane picture for a high-pass RC filter is very similar to the low-pass filter, except that this time we have both a pole (shown as an ×) and a zero (shown as an O), as shown in Fig. 20-9. The pole is at the same place as before, while the zero is at the origin.

At any particular frequency, the gain is the ratio B/D. For example, the pole at -1000 on the σaxis is for a first-order high-pass filter with a corner frequency of 1000 Hz. At a frequency of 2000 Hz, the distance B is 2000, while distance D is (as before)

$$D = \sqrt{1000^2 + 2000^2} = 2236$$

and so the gain is

$$\text{Gain} = \frac{B}{D} = \frac{2000}{2236} = 0.894$$

which is -0.969 dB, or approximately -1 dB, as we would expect one octave above the corner frequency.

Second Order filters

If you want skirts that are steeper than 6 dB per octave, you must go to a second- or higher-order filter. The simplest (though not the best) way to make a second-order filter is to just cascade two first-order RC filters as in Fig. 20-10; that is, to connect one after the other, so the signal must first go through one and then the second. Unfortunately, you cannot simply connect two identical filters together because they interfere with each other —
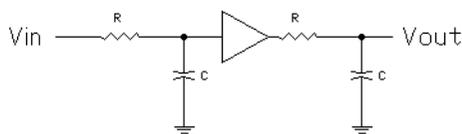

Fig. 20-10. A second-order RC filter

each one loads down the other. Short of some major redesign, the easiest solution is to insert an amplifier between the two so as to isolate one from the other, as shown in Fig. 20-10 (and ignore the question of how to properly design that amplifier).

Suppose the resistors are still 1590 ohms, and the capacitors 0.1 μF, as before, so the corner frequency of each filter is still 1000 Hz. This gives us two identical low-pass filters, and the resulting response is shown in Fig. 20-11. With two identical filters, the dB loss at any frequency is the sum of the two separate losses, so it simply doubles. As a result, the gain drops at 12 dB per octave, but also the gain at the cutoff frequency is now at -6 dB rather than -3 dB. This raises two small problems:

- Is the overall cutoff frequency still at 1000 Hz, or has it shifted down to about 5000 Hz, where the gain is now about -3 dB?
- What about the gain of the amplifier? Doesn't that increase the overall gain above 0 dB?

The second question is easier to answer — yes, the gain of the amplifier must be added to the filter gain. For example, if the amplifier gain is 12 dB, then the overall gain at low frequencies would be +12 dB, and it would drop to 0 dB and then below as the frequency rises. You would simply add 12 dB, the amplifier gain, to all the numbers at the left side of the frequency plot.

The first question is a bit tougher, however, so let's ignore it and look at the problem in a different way. In terms of poles, our circuit now contains two identical RC filter sections with each one having a pole, and so the circuit really has two poles, both at the same place as shown in Fig. 20-7. When using mathematics to de-
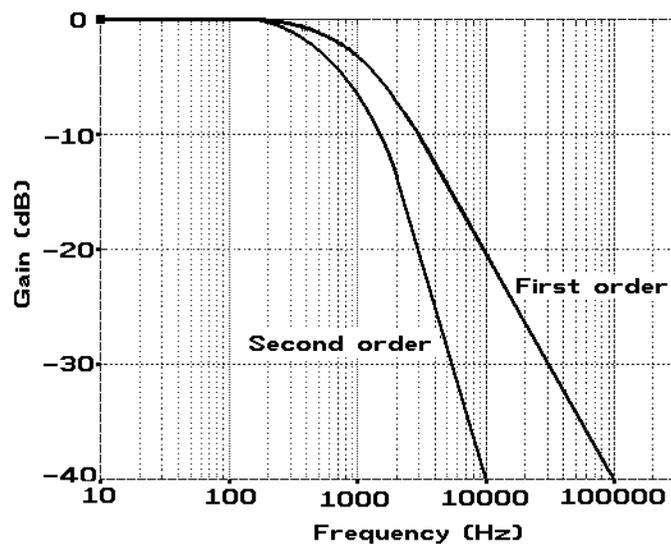

Fig. 20-11. Second order RC filter response

scribe this, engineers would say that there is a "double pole" at, in this case, $\sigma = -1000$.

It turns out to be much better to move the two poles apart, which leads us to a circuit called a Butterworth Filter.

Butterworth Filter

Fig. 20-12 shows the two poles for a *Butterworth* second-order filter. Instead of a double-pole at $\sigma = -1000$, the Butterworth circuit swings the poles 45° outward in a circle, as shown in the drawing. One goes up, the other down; they are still a distance of 1000 from the center, but now are at $\sigma = -707$, and at frequencies +707 (the higher one) and –707 (the lower one).

To calculate the gain at an arbitrary frequency, simply think of the two poles as being two separate filters, so that the gain is the product of the two separate gains. You therefore need to measure the two distances D_1 and D_2. As before, the gain due to one pole is A/D_1 and the gain due to the other pole is A/D_2, so the overall gain is

$$\text{Gain} = \frac{A}{D_1} \times \frac{A}{D_2} = \frac{A^2}{D_1\,D_2}$$

For example, to find the gain at 2000 Hz as shown in the drawing, we have

$$D_1 = \sqrt{(2000-707)^2 + 707^2} = 1474$$

$$D_1 = \sqrt{(2000+707)^2 + 707^2} = 2797$$

$$\text{Gain} = \frac{1000 \times 1000}{1474 \times 2797} = 0.24$$

which is -12.4 dB.

Fig. 20-13 compares the frequency response of the Butterworth filter with the normal double-pole response. You can see that the Butterworth filter has a sharper break at the corner frequency, and really does have –3 dB loss at the corner.
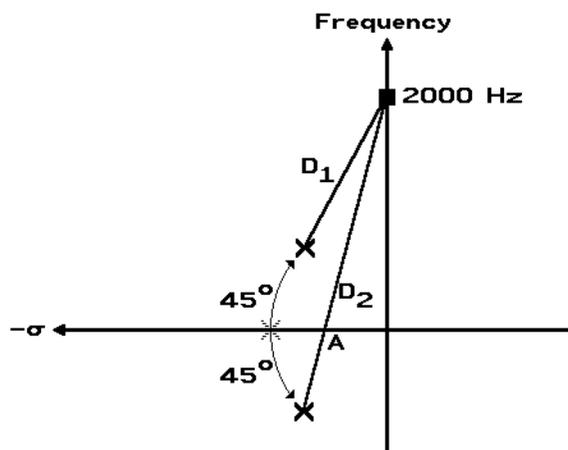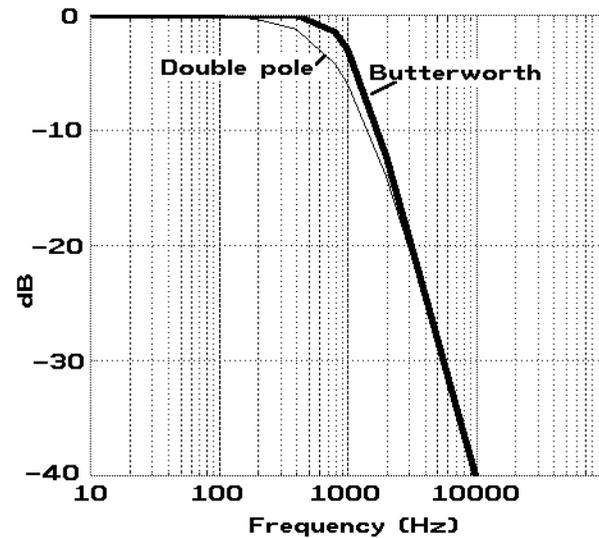

Fig. 20-12. Butterworth pole locations

Fig. 20-13. Butterworth low-pass filter response

The Butterworth filter is also called a "maximally flat" filter. The reason is that its top is the flattest that can be achieved. All other low-pass filters either have a more rounded top, or else have a top which is uneven and "lumpy".

The Chebyshev Filter

For some applications, the Chebyshev filter is better. Compare the Butterworth and Chebyshev filters in Fig. 20-14. Both second-order filters have the same 12 dB per octave skirt drop, but the Chebyshev filter starts off with a sharp rise and then a steeper drop, rather than gently curving like the Butterworth filter. The rise is called *overshoot*, and its height is measured in percent.
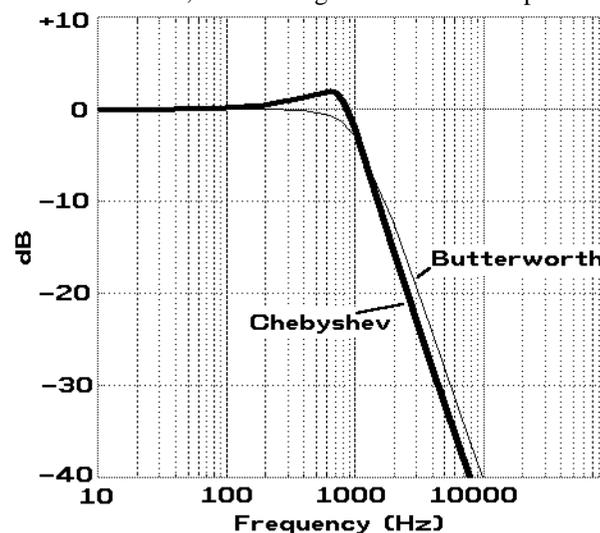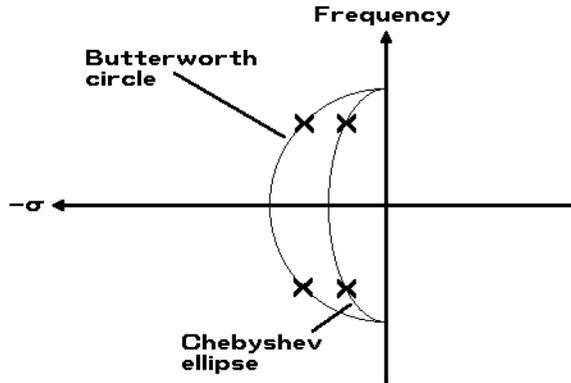

Fig. 20-14. Chebyshev filter with overshoot

Fig. 20-15. Butterworth and Chebyshev pole positions

Fig. 20-16. Higher-order Butterworth pole positions

At its peak, the gain of this particular filter has risen from 1 to about 1.25, which makes it about a 25% overshoot. When plotted in dB, the overshoot doesn't look very significant, since a gain of 1.25 is less than a 2 dB rise.

In the second-order filter, there is only one overshoot, but with higher-order filters there may be several; the result is then called *ripple*.

Fig. 20-15 shows how the pole positions determine the type of filter. In the Butterworth filter, the two poles are located on a circle, 45˚ above and below the horizontal. In a Chebyshev filter, the circle is collapsed into an ellipse. The flatter the ellipse, the more overshoot.

Higher-order filters

As we've seen so far, the number of poles determines the order of the filter. In a Butterworth filter, they are evenly spaced out along a circle; in a Chebyshev filter, they are spaced out along an ellipse. Fig. 20-16 shows several higher-order pole configurations for Butterworth filters. In each case, we see that the poles are evenly spaced out along the arc.

Passive vs. Active Filters

We have so far discussed some of the theoretical aspects of filters but, except for some very simple RC filters at the very beginning of this chapter, we have not shown any specific diagrams. In fact, higher-order filters are sometimes difficult to design — there are entire books devoted to just filter design.

Filters can be broken down into two basic types: passive filters and active filters.

- Passive filters use only passive components, which means resistors, capacitors, and inductors.
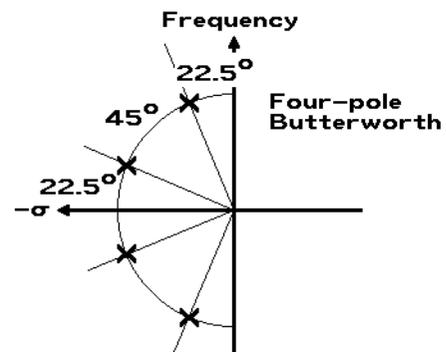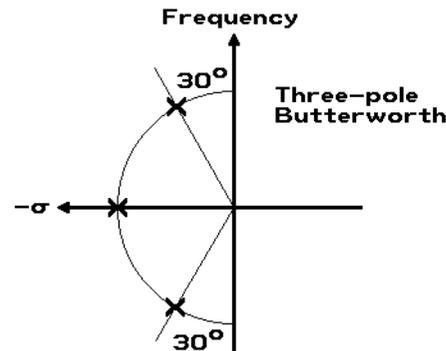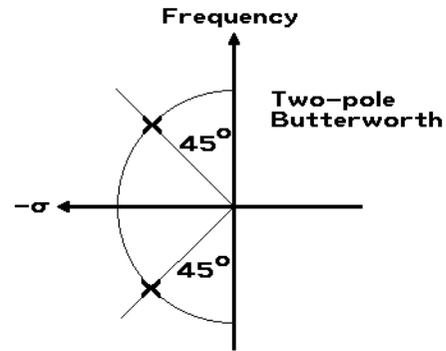- Active filters also use active components. which means amplifying devices such as transistors, in-tegrated circuits, or even vacuum tubes, in addition to passive components.

Each filter type has its advantages and disadvantages:

Passive filters, especially those with multiple poles, usually require inductors, which are sometimes big and expensive. Active filters can be built with just capacitors; since modern transistors and integrated circuits are cheap and small, the total cost, weight, and size can be much less.

Active filters, on the other hand, are limited by the active devices they use. One limit is the frequency range — passive components can work at much higher frequencies than ICs and transistors. Active devices also

have a limited dynamic range — signals that are too big get clipped, while signals that are too small may get buried in noise. This can be a severe problem if the purpose of the filter is to eliminate a strong interfering signal from a weak desired signal. The strong signal may cause clipping and distortion of both signals.

The general rule is therefore that low-frequency, low-power applications often use active filters, while high-frequency or high-power applications use passive filters.

Sallen-Key Low-Pass Active Filter Circuit

There are several popular active filter circuits; one popular one is the Sallen-Key low-pass circuit shown in Fig. 20-17. The same circuit can be used as a Butterworth or Chebyshev filter (or as a Bessel filter, another popular type) just by changing the component values. Mathematical analysis of the circuit is way beyond the level of this textbook, but very few people ever actually analyze the circuit. Most design is done by referring to tables and graphs in filter design books to find the component values to use.

The design procedure for a second-order Butterworth filter is as follows:

1. Choose the desired cutoff frequency f_C.

2. Choose a reasonable value for C_1. This involves some guesswork; if the value chosen is a bad one, the resistor values in the following steps may be too large or too small to be reasonable.

3. Choose the desired filter gain; possible values are 2, 4, 6, 8, or 10 (other values are possible, but not given in the following table.)

4. Calculate a number called Z:

$$Z = \frac{10^{-4}}{f_C \times C_1}$$

5. Calculate the remaining component values from the data in Table 20-1. If they don't seem reasonable, try a different value for C_1 and repeat.

Example 20-2. Low-Pass Active filter

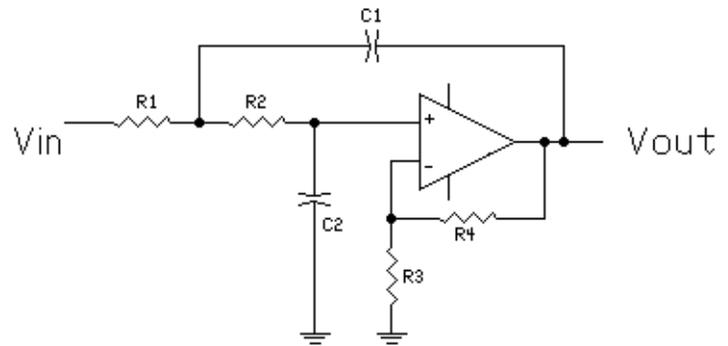Let's design a simple second-order active filter with a corner frequency of 1000 Hz, and a gain of 6, using the

Fig. 20-17. Sallen-Key active low-pass filter

data in Table 20-1. Start by choosing $C_1 = .01$ μF. The next step is to calculate Z from the equation

$$Z = \frac{10^{-4}}{f_C \times C_1} = \frac{10^{-4}}{10^3 \text{Hz} \times 10^{-8} \text{ Farad}} = 10$$

From the Gain=6 column of Table 20-1, the remaining components are

$$R1 = 620Z = 6200 \text{ ohms}$$
$$R2 = 2100Z = 21 \text{ Kohms}$$
$$R3 = 3200Z = 32 \text{ Kohms}$$
$$R4 = 16000Z = 160 \text{ K ohms}$$
$$C2 = 2 \ C1 = 0.02 \text{ μF}$$

The various resistor values are reasonable for op-amp circuits, so the initial choice of capacitor C1 is acceptable.

Sallen-Key high-pass active filter

The diagram for a Sallen-Key active high-pass filter is almost identical with the low-pass filter, except that R_1 and R_2 get replaced by C_1 and C_2, while C_1 and C_2 are replaced by R_1 and R_2. The formulas for component values are, however, different. The best place for design information is one of the active filter design books.

TABLE 20-1. Sallen-Key Butterworth active filter					
Gain	2	4	6	8	10
R_1	1120Z	700Z	620Z	530Z	470Z
R_2	2250Z	3800Z	2100Z	2500Z	2800Z
R_3	6900Z	5800Z	3200Z	3400Z	3600Z
R_4	6900Z	17500Z	16000Z	24000Z	32000Z
C_2	C_1	C_1	$2\times C_1$	$2\times C_1$	$2\times C_1$