

## Chapter 21

### Band-Pass Filters and Resonance

In Chapter 20, we discussed low-pass and high-pass filters. The simplest such filters use RC components — resistors and capacitors. It is also possible to use resistors and inductors to make RL filters. Their equations are similar, but we normally try to avoid inductors wherever possible, so RL filters are not as common.

We point out that such filters have a corner or cutoff frequency, which is the frequency at which  $R = X$ . We will see something similar here as well.

#### LC Circuits

The simplest band-pass filters are LC filters — they use inductors and capacitors (though there is always some extra resistance in the circuit which affects operation.) These components can be connected in series or parallel with each other; the resulting circuits are called *series resonant* or *parallel resonant* circuits, respectively. The word resonant is used because these circuits respond to particular frequencies, much like the strings on a violin or guitar. For that reason, they are also often called *tuned* circuits.

Fig. 21-1 shows some examples of these circuits connected in different ways. Both the series as well as the parallel circuit have a *resonant frequency* which (neglecting the effects of resistance) is the frequency at which the  $X_L$  of the inductor is equal to the  $X_C$  of the capacitor. We can find this resonant frequency by solving the equation

$$X_L = X_C$$

$$2\pi fL = \frac{1}{2\pi fC}$$

where  $f$  is the frequency,  $L$  is the inductance, and  $C$  is the capacitance, for the frequency:

$$f_{resonant} = \frac{1}{2\pi\sqrt{LC}}$$

The operation of both circuits depends on the fact that the voltage and current in inductors and capacitors are  $90^\circ$  out of phase. You may remember “ELI the ICE man” — this little phrase reminds us that the voltage (E) comes  $90^\circ$  before the current (I) in the inductor (the current *lags* behind the voltage), but  $90^\circ$  after the current in the capacitor (the current *leads* the voltage).

In the parallel-tuned circuit, the capacitor and inductor are in parallel, and they therefore have the same voltage. At the resonant frequency we also have  $X_L = X_C$ ; their reactances are equal. Since Ohm’s Law for ac circuits says that  $V = IX$ , if the inductor and ca-

pacitor both have the same voltage and the same reactance, they must also have the same current. and so they both have the same current. But because one of these currents leads the voltage by  $90^\circ$ , whereas the other current lags the voltage by  $90^\circ$ , they are  $180^\circ$  apart. The currents therefore go in opposite directions — when one goes up, the other goes down. Hence the current in the wire which leads to the parallel-tuned circuit must be zero. Since the external current into the tuned circuit is zero, the circuit behaves like an open circuit (which also has a voltage but no current through it.)

The opposite happens in the series-tuned circuit. Here, both the capacitor and the inductor have the same current since they are in series. This time, the voltage across one of them leads the current, while the voltage in the other lags the current by  $90^\circ$ . The two voltages are therefore  $180^\circ$  apart. At resonance (which is another way of saying “at the resonant frequency”), their reactances are equal and so their voltages are equal, but opposite. The total voltage across the series circuit is therefore zero, even though there is a current through it. The circuit therefore behaves like a short circuit (which also has a current but no voltage across it.)

We therefore form the following rules of thumb:

- At resonance, a parallel-tuned circuit behaves like an open circuit.
- At resonance, a series-tuned circuit behaves like a short.
- At other frequencies, both circuits have some impedance. Close to the resonant frequency, the circuits are not quite an open (for the parallel-tuned) or short (for the series-tuned), but still fairly close to it. The farther we go away from the resonant frequency, the less the circuits behave like an open or short circuit.

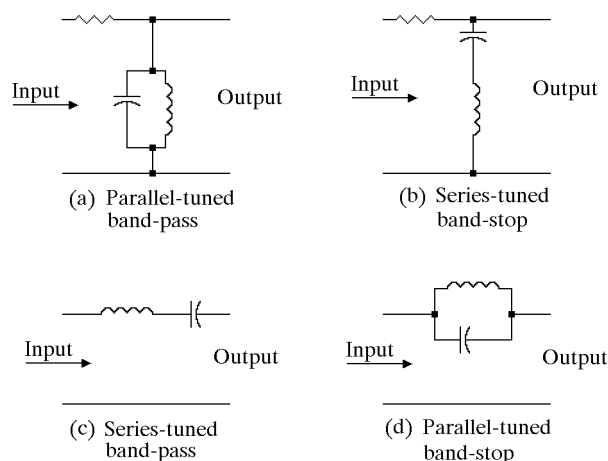


Fig. 21-1. Parallel and series resonant circuits

Both of these circuits can be used as selective filters to let some frequencies get through, and stop others. They can be connected in one of two basic ways — either the tuned circuit can be connected between a signal and ground as in Fig. 21-1 (a) and (b) (in which case it will short some of the signal to ground, depending on the circuit’s impedance), or so that a signal has to travel through the LC circuit to get from the input to the output as shown in (c) and (d) (in which case more or less will get through, again depending on the circuit’s impedance).

Consider circuit (a), for example. At resonance, the parallel-tuned circuit behaves like an open circuit, and most of the input signal travels right through the resistor to the output. Depending on the load at the output, there may be some current in the series resistor and so there may be some voltage loss, but we will ignore this. Away from resonance, however, the tuned circuit is no longer open; it causes an increased current to flow through the series resistor, and the voltage drop therefore increases. The farther away from resonance, the greater this drop, and the smaller the output voltage.

If we keep the input voltage constant but vary the frequency, and then plot the output voltage vs. frequency, we get a plot similar to Fig. 21-2 (a). We see that the peak in output occurs at the resonant frequency, and there is a dropoff on both sides. There is a band of frequencies around resonance that do get through, while frequencies far away from resonance are reduced (though not entirely stopped). Since there is this band of frequencies that get through, this is called a *band-pass* filter. Circuit (c) in Fig. 21-1 is also a band-pass filter; since the series-tuned circuit is a short circuit at resonance, frequencies at (and near) resonance get through, while frequencies farther away are reduced because the series-tuned circuit now has some reactance.

Circuits (b) and (d) do the opposite — at resonance, they stop the signal; circuit (b) does it by shorting the signal, whereas circuit (d) does it by opening the path between the input and output. Even near resonance they

reduce the signal, so they stop (or reduce) a band of frequencies, as shown in Fig. 21-2 (b). They are therefore called *band-stop* or *band-reject* filters.

Notice a similarity here with the RC low-pass or high-pass filters of the previous chapter. Although we there referred to a cutoff frequency, the actual curves showed that the “cutoff” was actually a very gradual drop. Here too we have a gradual change on both sides of the resonant frequency.

## Bandwidth

Returning to Fig. 21-2 (a), we’re interested in measuring how wide a band of frequencies gets through a band-pass filter; that is, we want to know the *bandwidth*. Clearly the width of the curve depends on where you measure it; the customary point is to measure the bandwidth at the point where the height of the curve is 70% of the maximum height; this is labelled BW in Fig. 21-2. (To be exact, the amplitude is  $1/\sqrt{2}$  or 0.707 of the maximum.) This point also happens to be 3 dB below the maximum, as we can see from

$$20 \log_{10} \frac{0.707}{1} = 20 \times (-0.15) = -3 \text{ dB}$$

and so it is often called the “–3 dB point” or “half-power point”. The left side –3 dB point is called the *lower 3 dB point*, while the other is the *upper 3 dB point*. Note that the minus sign is often skipped; it’s understood that there is a loss there and so it must be minus.

## Quality or “Q”

The narrower the bandwidth, the “better” the circuit, and so we define the *Q* or *Quality* of a tuned circuit as

$$Q = \frac{\text{resonant frequency}}{\text{bandwidth}}$$

For example, if a circuit resonant at 1 MHz has a bandwidth of 50 kHz (at the –3 dB point), then the *Q* would be 20; if the bandwidth is only 40 kHz, then the *Q* would be 25, which would be considered “better” or “higher quality” for some applications. Fig. 21-3 shows how the *Q* affects the response and bandwidth of a tuned circuit — higher *Q* means steeper skirts.

The *Q*, in turn, depends on the resistance in the circuit. Ideally, a tuned circuit would consist of only capacitance and inductance; with no added resistance; the *Q* would be infinite because the circuit would be ideal. In practice, however, there is always some resistance in a circuit, and this degrades the quality.

The added resistance could be in one of two places: it could be in series with the inductor (for instance, every inductor consists of wire that has some resis-

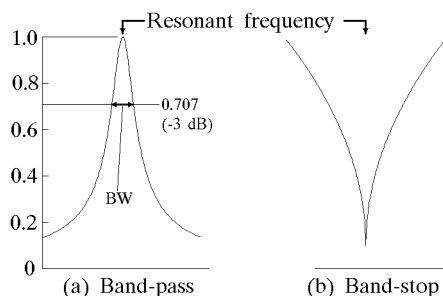


Fig. 21-2. Band-pass and band-reject frequency responses

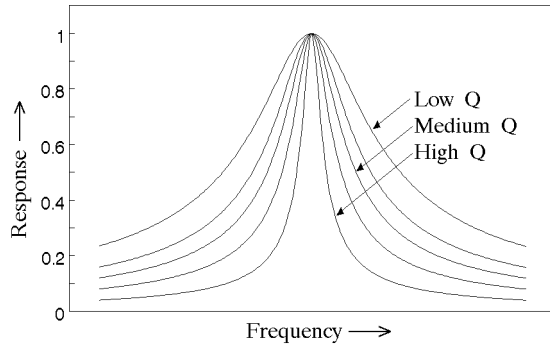


Fig. 21-3. Effect of Q on the bandwidth

tance), or it could be in parallel with it (such as the resistance of whatever other circuitry the tuned circuit is connected to). If  $X_L$  is the reactance of the inductor at the resonant frequency, then the Q is defined as

$$Q_{series} = \frac{X_L}{R_{series}}$$

if the resistance is in series with the inductor, and it is

$$Q_{parallel} = \frac{R_{parallel}}{X_L}$$

if the resistance is in parallel. Note that, since R and  $X_L$  are both in ohms, Q has no units.

In most cases, there are resistances both in series and in parallel with the inductor. In that case, we need to take both effects into our calculations. There are some fairly complex equations used in circuit theory which give us the total Q, but a much easier way of finding the equivalent actual Q due to both resistances is

$$Q_{actual} = \frac{Q_{series} Q_{parallel}}{Q_{series} + Q_{parallel}}$$

You may recognize that this is the same format as the “product over the sum” formula for parallel resistors.

### Example 21-1. Q and Bandwidth

Let’s do a sample problem: what is the bandwidth for the circuit in Fig. 21-4? First, find the resonant frequency:

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2 \times 3.14159 \times \sqrt{10^{-3} \times 10^{-6}}} = 5033 \text{ Hz}$$

Actually, if we used the full equation that many textbooks give for the resonant frequency when there is a resistor inside the tuned circuit, we would find that the resonant frequency is about 5038 Hz. But the difference is only about 1/10%, and so our simpler equation is perfectly adequate.

Next, find  $X_L$  at this frequency:

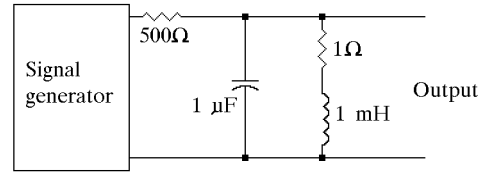


Fig. 21-4. Example for calculating the Q and bandwidth

$$X_L = 2\pi fL = 2 \times 3.14159 \times 5033 \times 10^{-3} = 31.623 \Omega$$

There is a 1-ohm resistor in series with the inductor, so the series Q is

$$Q_{series} = \frac{X_L}{R_{series}} = \frac{31.623}{1} = 31.623$$

What about the parallel resistance? If we assume that the signal generator has zero output resistance and imagine that we are looking outward from the inductor at whatever resistances are outside, we would see the 500-ohm resistor in parallel with the inductor. (We could prove that more rigorously by applying Thevenin’s theorem). Hence we use 500 ohms in the parallel Q formula:

$$Q_{parallel} = \frac{R_{parallel}}{X_L} = \frac{500}{31.623} = 15.811$$

The actual Q is therefore

$$Q_{actual} = \frac{Q_{series} Q_{parallel}}{Q_{series} + Q_{parallel}} = \frac{31.623 \times 15.811}{31.623 + 15.811} = 10.54$$

Note how the actual Q is a combination of the series and parallel Q. The fact that the equation has the same form as the “product over the sum” equation for parallel resistors means that we can apply similar reasoning to the Q as we can to parallel resistors. That is,

- When two resistors are in parallel, the total parallel resistance is always smaller than the smaller resistor. Likewise, the actual Q is always smaller than either the parallel Q or the series Q.
- If one of two parallel resistors is much larger than the other, we can approximate the total parallel resistance by ignoring the larger resistor. The same applies to the Q. For example, if the series and parallel Q were 5 and 50, the actual Q would be 4.55, which is very close to 5, the smaller Q. Hence it is the smaller Q that plays a major role in setting the actual Q.

We now find the bandwidth from

$$Q = \frac{\text{resonant frequency}}{\text{bandwidth}}$$

$$\text{bandwidth} = \frac{\text{resonant frequency}}{Q} = \frac{5033 \text{ Hz}}{10.54} = 477 \text{ Hz}$$

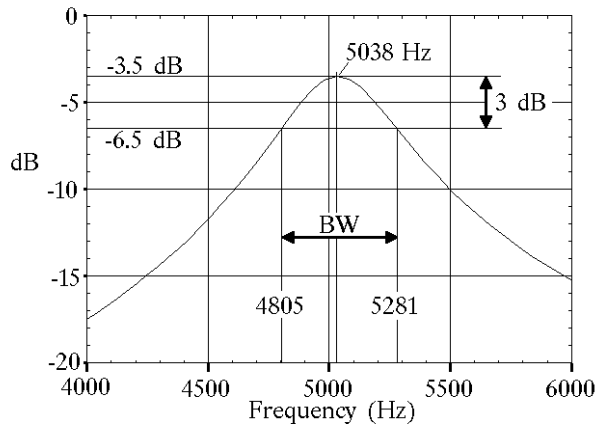


Fig. 21-5. Frequency Response for Example 21-1

Fig. 21-5 shows the actual bandpass for this circuit, with the gain plotted in dB. At the peak (which occurs at the actual resonant frequency of 5038 Hz) there is a loss of approximately 3.5 dB in the circuit, so the peak plots at -3.5 dB.

The measured resonant frequency is 5038 Hz, the lower 3-dB frequency (the lower frequency at which the gain drops by 3 dB, from -3.5 dB down to -6.5 dB) is 4805 Hz, and the upper 3-dB frequency is 5281 Hz. The measured bandwidth is thus

$$5281 \text{ Hz} - 4805 \text{ Hz} = 476 \text{ Hz}$$

which is quite close to the calculated value.

Incidentally, the frequency scale in Fig. 21-5 is linear; this is fairly common when only a small range of frequencies is plotted; when plotting large spans of frequencies, however, we generally use a logarithmic frequency scale, as shown in Fig. 21-6.

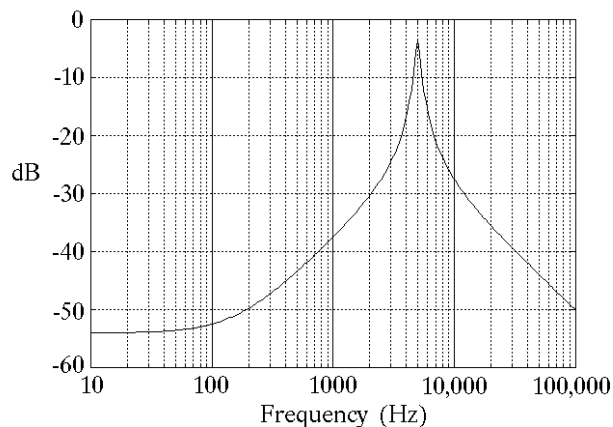


Fig. 21-6. More complete frequency response for the example

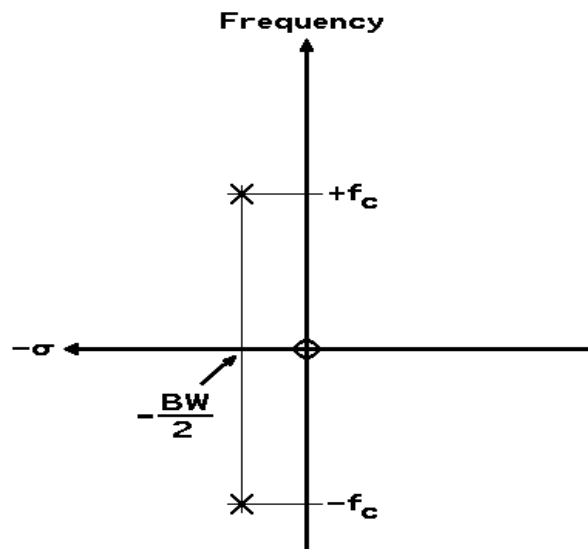


Fig. 21-7. Zero and poles for band-pass filter

## Poles and Zeroes

Now that we have learned some basic ideas about band-pass filters, let's return to s-plane poles and zeroes as we discussed in the previous chapter.

In a way, we can think of the bandpass filter as a combination of a high-pass filter (which removes the low frequencies) and a low-pass filter (which removes the high frequencies.) The simplest low-pass filters are first-order and have one pole; the simplest high-pass filters are also first-order and have one pole and one zero. The simplest band-pass filter, however, is a second-order filter and has two poles and one zero.

These are located as shown in Fig. 21-7. The zero is at the origin; one pole is up at a distance equal to the resonant frequency and left a distance equal to half of the bandwidth BW; the other pole is the same distance to the left, but down a distance equal to the resonant frequency. To find the gain at any frequency, plot that frequency up on the frequency axis, and then calculate G as

$$\text{Gain} = BW \times \frac{\text{distance to the zero}}{\text{distance to one pole} \times \text{distance to other pole}}$$

where BW is the 3-dB bandwidth. This forces the gain at the resonant frequency to be exactly 1; that is, 0 dB.

## Example 21-2. Second order band-pass

Let's repeat the circuit of Fig. 21-4, using the data from Example 2-1. Since the resonant frequency was found as 5033 Hz, we plot the two poles a distance of 5033 up and down. The bandwidth was 477 Hz, so the poles are

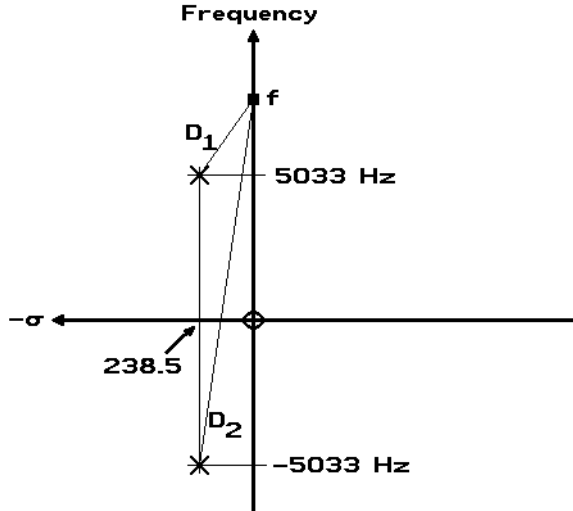


Fig. 21-8. Poles and zeroes for Example 21-2

plotted a distance of  $477/2$  or 238.5 to the left. The zero is at the origin, as shown in Fig. 21-8 (except that the figure is not drawn exactly to scale).

Let's now say that we wish to calculate the gain at some frequency  $f$ , say, 6000 Hz. The distance from the point  $f$  to the zero is just equal to the frequency  $f$ . The distance  $D_1$  to the top pole is

$$D_1 = \sqrt{(f - 5033)^2 + 238.5^2}$$

$$= \sqrt{967^2 + 238.5^2} = 995.98$$

The distance  $D_2$  from the point  $f$  to the bottom pole is

$$D_2 = \sqrt{(f + 5033)^2 + 238.5^2}$$

$$= \sqrt{11033^2 + 238.5^2} = 11035.58$$

Finally,

$$\text{Gain} = BW \times \frac{\text{distance to the zero}}{\text{distance to one pole} \times \text{distance to other pole}}$$

$$= 477 \times \frac{6000}{995.98 \times 11035.58} = 0.260$$

which converts to -11.7 dB.

Compare this with Fig. 21-5, which says that the gain for this circuit at 6000 Hz is roughly -15.2 dB — why the difference of about 3.5 dB?

Simple. The poles and zeroes don't know the exact structure of the circuit, and so don't know that the two resistors in the circuit (see Fig. 21-4) are losing some signal; thus the gain at the resonant frequency is -3.5 dB, as shown in Fig. 21-5, not 0 dB. Thus *all* the gains

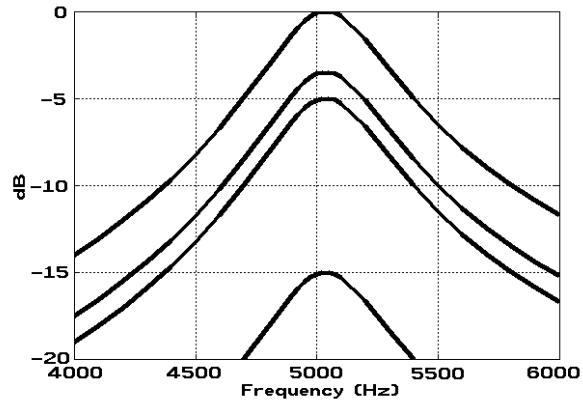


Fig. 21-9. Identical response curves

are really 3.5 dB less than the pole-zero plots would show.

Fig. 21-9 illustrates the point. All of the curves in the figure look the same and describe the same frequency response. The only difference is that some are higher than others, meaning that there are different amounts of gain or loss in the circuit, but these gain differences do not affect the basic frequency response.

### Higher-order bandpass filters

With low- or high-pass filters, we went to higher orders to get steeper skirts, and greater attenuation above (or below) the corner frequency. With band-pass circuits, we go to higher orders for similar reasons.

With a band-pass filter, you can get steeper skirts close to the center frequency just by going to a higher  $Q$ , as shown in Fig. 21-3. Unfortunately, steeper skirts and greater attenuation also narrow the bandwidth, which is often bad, and the higher  $Q$  does not improve the slope far away from resonance. A higher-order filter lets us get both a sharp filter with steep skirts, and also (if we wish) a larger bandwidth.

Fig. 21-10 shows how to get a larger bandwidth. We take several second-order band-pass filters and *stagger-*

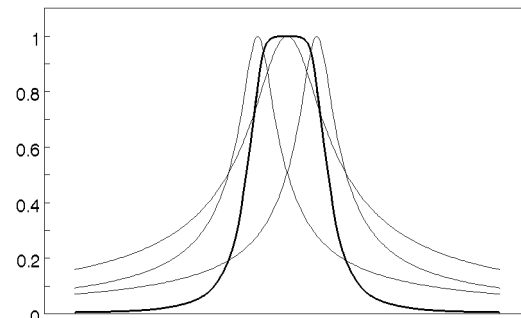


Fig. 21-10. Stagger tuning for greater bandwidth

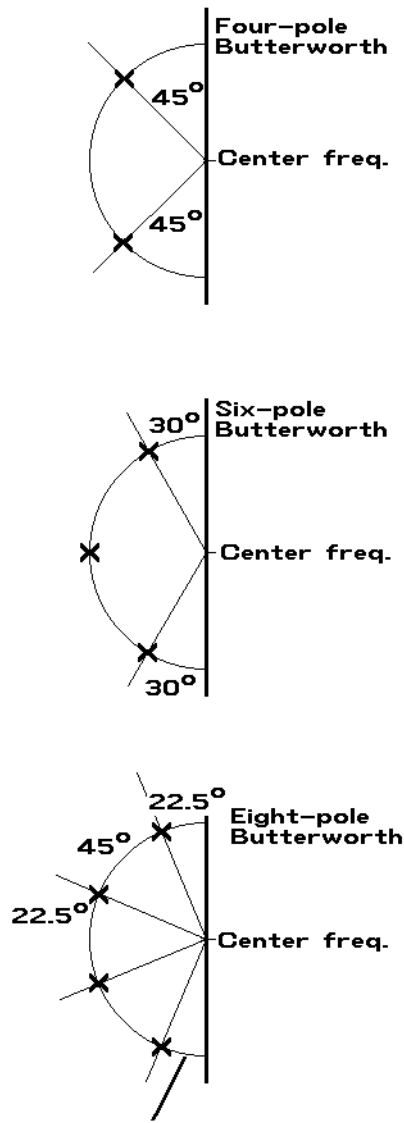


Fig. 21-11. Higher-order band-pass filter pole positions

tune them; that is, tune them all to slightly different frequencies. Sending the signal through all three filters, one after another, gives us a sixth-order filter (the dark curve) with a wider bandwidth and a flatter top. However, it's important to carefully adjust each individual filter so it is resonant at the correct frequency and has the correct Q; otherwise the result may not be what we want.

As in the previous chapter, we can select various kinds of filters. A Butterworth filter will be maximally flat — the top of the pass band will be as smooth as possible. A Chebyshev filter may have some ripple at the

top, but will have sharper sides. A Bessel or other kind of filter may be good for some other reason.

To select the correct center frequency and Q for each filter, we must decide where to put its poles. We do it very much like we did in Chapter 20 — by placing the poles on circles or ellipses, as in Fig. 21-11. To start, you locate the center frequency of the desired filter on the frequency axis. Two sets of poles are needed — one above the origin and one below, since each filter has a pair of poles, one on top, the other on bottom. Fig. 21-11 only shows the top sets of poles, but the bottom is a mirror image.

Then you draw a circle at that point; the circle diameter should equal the overall desired bandwidth (you would use an ellipse for a Chebyshev filter; the flatter the ellipse, the more ripple.)

Then you position the poles evenly around the circle, as shown. For each pole, there is a matching pole “down under”, plus a zero at the origin. For each pole (and its match down under and its zero) there will be a single second-order band-pass filter circuit (called a *stage*), with the signal having to pass from one filter stage to the next.

As before, each filter stage can be either an active filter or a passive filter. But, again as before, passive filter stages cannot be connected one right after another without special care; the safest is to separate them by an amplifier.

### Active Band-Pass filters

Just as with low-pass or high-pass filters, band-pass filters can be designed with op-amps to make an active filter. Fig. 21-12 shows a popular circuit with one op-amp which makes a second-order active band-pass filter. The component values are easy to find from the following equations. Given the resonant frequency  $f$ , the desired Q, and the desired gain G, choose a convenient value for the capacitors C. For typical audio filters, common values are either 0.1  $\mu F$  or 0.01  $\mu F$ . Then

$$R_1 = \frac{Q}{2\pi fGC}$$

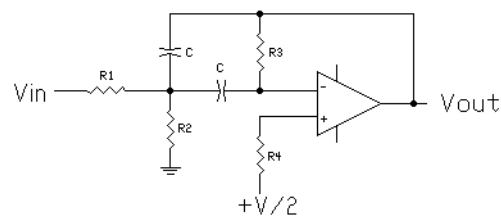


Fig. 21-12. Active second-order bandpass filter

$$R_2 = \frac{R_1 G}{2Q^2 - G}$$

$$R_3 = R_4 = 2GR_1$$

Since capacitors are seldom the exact values they are labelled, resistors  $R_2$  and  $R_3$  are usually made somewhat larger and variable, so the center frequency and  $Q$  can be adjusted to the exact desired values.

### Example 21-3. Fourth-order Butterworth active band-pass filter

Assignment: Design a four-pole active Butterworth filter centered at 5000 Hz and having a bandpass of 500 Hz. Since the circuit of Fig. 21-12 is only a two-pole filter, we will need two stagger-tuned cascaded stages.

Fig. 21-13 shows the first step — positioning the poles. We will use a total of four poles and two zeroes, but only the top two poles need to be shown, since we know that the other two poles (down below) are mirror images of these two, and the two zeroes are both at the origin.

We first draw a semicircle with its center at 5000 Hz, and with a diameter equal to the 500 Hz bandwidth. Now place the two poles at  $45^\circ$  angles as shown.

The two poles are at the corners of  $45^\circ$  triangles whose hypotenuse is 250 (the radius of the circle, which is half of the bandwidth). We know that the short sides of a  $45^\circ$  triangle are  $1/\sqrt{2}$  of the hypotenuse, which is  $250/\sqrt{2} = 177$ . This places the two poles at heights of

$$5000 + 177 = 5177 \text{ Hertz}$$

and

$$5000 - 177 = 4823 \text{ Hertz}$$

and 177 to the left of the frequency axis. We now have the poles for the two second-order filter stages.

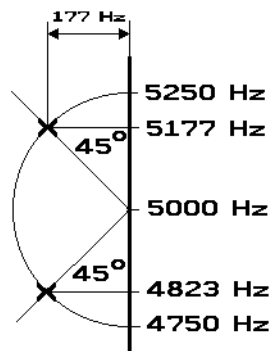


Fig. 21-13. Poles for Example 21-3

The first stage has poles at  $(-177, 5177)$ . The center frequency of this stage is thus 5177 Hz. Since the distance to the left ( $-177$ ) is half the bandwidth, the bandwidth is 354 Hz, and so the  $Q$  is  $5177/354$  or 14.6.

The second stage has poles at  $(-177, 4823)$ . The center frequency of this stage is thus 4823 Hz, the bandwidth is also 354 Hz, and the  $Q$  is  $4823/354$  or 13.6.

Let's use a  $0.01 \mu F$  capacitor and design for a gain  $G$  of 10 per stage. For the first stage,

$$R_1 = \frac{Q}{2\pi f G C} = \frac{14.6}{2\pi \times 5177 \times 10 \times 10^{-8}} = 4488 \text{ ohms}$$

$$R_2 = \frac{R_1 G}{2Q^2 - G} = \frac{4488 \times 10}{2 \times 14.6^2 - 10} = 107.8 \text{ ohms}$$

$$R_3 = 2GR_1 = 2 \times 10 \times 4488 = 89.76 \text{ Kohms}$$

For the second stage,

$$R_1 = \frac{Q}{2\pi f G C} = \frac{13.6}{2\pi \times 4823 \times 10 \times 10^{-8}} = 4488 \text{ ohms}$$

$$R_2 = \frac{R_1 G}{2Q^2 - G} = \frac{4488 \times 10}{2 \times 13.6^2 - 10} = 124.7 \text{ ohms}$$

$$R_3 = 2GR_1 = 2 \times 10 \times 4488 = 89.76 \text{ Kohms}$$

Fig. 21-14 shows the frequency response of the two individual stages (light lines) and the combined circuit with the two stages in cascade (heavy line) as calculated from the pole-zero positions; Fig. 21-15 shows the frequency response of the actual circuit using the above components.

Note that the response in Fig. 21-14 is not quite symmetrical. There are several reasons. Partially, there are always some slight rounding errors in doing calculations; the results might look a bit better if we carried all

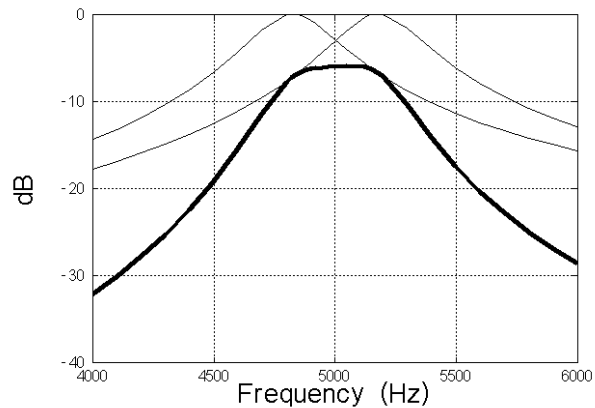


Fig. 21-14. Response for Example 21-3, from poles and zeroes

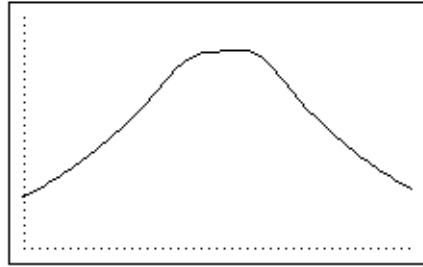


Fig. 21-15. Response of Example 21-3, from circuit values

answers out to eight, ten, or even more decimal places. Thus, due to slight approximations, the actual center frequency of the plot in Fig. 21-14 is 5006 Hz, not 5000 Hz.

A second reason is that Fig. 21-14 is plotted on regular linear graph paper; the curve would look more symmetrical on the logarithmic paper that is normally used for frequency response curves. Let's look into that a bit more:

With the actual center frequency at 5006 Hz and a 500 Hz bandwidth, we would expect the  $-3$  dB points to be at  $(5006 - 250) = 4756$  Hz, and at  $(5006 + 250) = 5256$  Hz. But they are actually at 4763 and 5262 Hz, because on logarithmic graph paper, 5006 would be exactly centered between 4763 and 5262 Hz. That is, on log graph paper, equal ratios occupy equal space — the ratio  $5006/4763$ , which defines the distance from the center frequency to the lower  $-3$  dB frequency, is the same as the ratio  $5262/5006$ , which defines the equal opposite distance. We then have

$$\frac{5006}{4763} = \frac{5262}{5006}$$

$$5006 \times 5006 = 5006^2 = 4763 \times 5262$$

and therefore

$$5006 = \sqrt{4763 \times 5262}$$

Thus the center frequency is not at the arithmetic mean (average or half-way between) the  $-3$  dB frequencies, but at the *geometric mean*.

We use normal graph paper rather than logarithmic graph paper when plotting relatively small frequency differences because commercial log paper is only available with 4 or more cycles on the page. A small frequency difference, which often takes just a fraction of one cycle, would be too small to see on the graph. Fig. 21-16 shows how the data from Fig. 21-14 would look on such a graph.

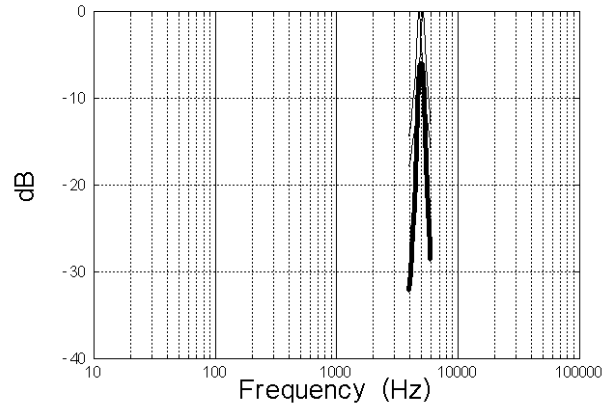


Fig. 21-16. Data from Fig. 21-14 on regular log paper